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ALMOST ISOMETRIC EMBEDDING BETWEEN METRIC SPACES

BY

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ABSTRACT

We investigate the relations of *almost isometric embedding* and of *almost isometry* between metric spaces.

These relations have several appealing features. For example, all isomorphism types of countable dense subsets of $\mathbb{R}$ form exactly one almost-isometry class, and similarly with countable dense subsets of Uryson's universal separable metric space $\mathbb{U}$.

We investigate geometric, set-theoretic and model-theoretic aspects of almost isometry and of almost isometric embedding.

The main results show that almost isometric embeddability behaves in the category of *separable* metric spaces differently than in the category of general metric spaces. While in the category of general metric spaces the behavior of universality resembles that in the category of linear orderings — namely, no universal structure can exist on a regular $\lambda > \aleph_1$ below the continuum — in the category of separable metric spaces universality

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behaves more like that in the category of graphs, that is, a small number of metric separable metric spaces on an uncountable regular $\lambda < 2^{\aleph_0}$ may consistently almost isometrically embed all separable metric spaces on λ.

1. Introduction

In this paper we investigate two relations between metric spaces: *almost isometric embedding* and *almost isometry.*

The study of these relations is motivated by three different considerations, the first of which is geometric. From a geometric point of view, the relations of isometry and of isometric embedding between metric spaces are too strong: the metric space $\pi\mathbb{Q}$ is not isometric to $\mathbb{Q}$ and no subinterval of $\pi\mathbb{Q}$ can be isometrically embedded into $\mathbb{Q}$, although both spaces "look the same" geometrically. On the other hand, the relations of *bi-Lipschitz embeddability* and of *bi-Lipschitz homeomorphism* are too weak, since, given a Lipschitz constant $K > 1$, the graph of $y = \alpha \sin x$ for small $\alpha > 0$ and $\mathbb{R}$ are K-bi-Lipschitz homeomorphic, though geometrically different.

The relations we study are strictly weaker than their corresponding isometry relations and strictly stronger than their corresponding bi-Lipschitz ones. Almost isometry is sufficiently weak to make all countable dense subsets of $\mathbb{R}$ almost isometric to each other, yet sufficiently strong so that only isomorphism types of countable dense subsets of $\mathbb{R}$ are almost isometric to $\mathbb{Q}$ — thus all isomorphism types of countable dense subsets of $\mathbb{R}$ form exactly one almost isometry class.

Our second motivation is set-theoretic. Up to *order-isomorphism* there is just one countable dense subset of $\mathbb{R}$, and Baumgartner [1] proved long ago that it is consistent with the axioms of set theory that up to order isomorphism there is just one $\aleph_1$-dense subset of $\mathbb{R}$. Also up to *almost isometry* there is just one countable dense subset of $\mathbb{R}$. Does the analog of Baumgartner's consistency hold for almost isometries? Namely, is it consistent with the axioms of set theory that all $\aleph_1$-dense subsets of $\mathbb{R}$ are almost isometric to each other? Such a consistency would be a strengthening of Baumgartner's consistency, since an almost isometry between two dense subsets of $\mathbb{R}$ implies they are either order isomorphic or order anti-isomorphic, and there is some $\aleph_1$-dense subset of $\mathbb{R}$ which is order isomorphic to its inverse.

Furthermore, in the category of separable metric spaces, Uryson's universal separable metric space $\mathbb{U}$ plays the exact role that $\mathbb{R}$ plays in the category of separable linear orderings. Just as $\mathbb{R}$ is the unique separable complete, ultrahomogeneous and universal separable linear ordering, $\mathbb{U}$ is the unique separable, complete and isometry-ultrahomogeneous universal metric space. It is natural to to ask about the relation of almost isometry between $\aleph_1$-dense subsets of $\mathbb{U}$.

We shall see below, though, that the analog for almost-isometries of Baumgartner's consistent statement about $\aleph_1$-dense real order types is too strong to be consistent with the axioms of set theory. Not only in $\mathbb{R}$ but also in the Uryson space $\mathbb{U}$ (see below) and in many other separable spaces there are always many pair-wise bi-Lipschitz incomparable $\aleph_1$-dense subsets.

Our third and final consideration is model-theoretic. An important problem in model theory is to determine, for a class of structures, the *universality spectrum* of the class: the class of infinite cardinals in which the class of structures has a universal structure. When adopting almost isometric embedding and almost isometries as the notions of embeddability and similarity, the class of metric spaces has a unique countable and homogeneous universal model. We study below the universality spectrum of the class of separable metric spaces and the class of general metric spaces in uncountable cardinalities. The universality spectrum of metric spaces has several similar properties to universality spectra in some of the other classes the authors have looked at — linear orders, infinite graphs, Abelian groups and models of a stable unsuperstable first-order theory, but is not quite the same as any one of them. The main results below show the universality behavior in the class of separable spaces is quite different than that in the class of general metric spaces.

1.1. Organization of the paper. The definitions of almost isometry and of almost isometric embedding and their basic properties are presented and discussed briefly in Section 2. In Section 3 we examine two properties of separable metric spaces: uniqueness of countable dense subsets up to almost isometry and almost isometric uniqueness of the whole space. It turns out that both properties are satisfied by many well-known metric spaces.

In Section 4 we investigate almost isometric embeddability on $\aleph_1$-dense subsets of $\mathbb{R}^d$ and on $\aleph_1$-dense subsets of Uryson's separable universal space $\mathbb{U}$. We show that in each case there are $2^{\aleph_0}$ pair-wise non-bi-Lipschitz-embeddable $\aleph_1$-dense subsets. In the case of $\mathbb{U}$ this means that there are continuum many pair-wise non-bi-Lipschitz-embeddable separable metric spaces, each of cardinality $\aleph_1$.

Section 5 presents universality results. It is first shown that fewer than continuum λ-dense subsets of $\mathbb{U}$ consistently suffice to almost isometrically embed every λ-dense subset of $\mathbb{U}$, for a regular uncountable $\lambda < 2^{\aleph_0}$. This is not as strong as proving the consistency of the existence of a single universal metric space on λ below the continuum, but it makes a partial analog to the consistency of a universal graph in λ below the continuum [14, 15, 16].

This consistency is complemented by the negative result in Section 5.2 about the category of general metric spaces. We prove that, in contrast to the separable case, it is *not* possible to have fewer than continuum metric spaces of cardinality λ so that every metric space on λ is almost isometrically embedded into one of them when the continuum is larger than λ and $\lambda > \aleph_1$ is regular. The method for proving this negative result is the method of attaching invariants to structures modulo club guessing ideals, which was introduced in [10] for linear orderings and was later used for classes of models of stable-unsuperstable theories, classes of infinite Abelian groups and certain infinite graphs [10, 11, 12, 9, 5].

The results in Sections 3 and 5.2 were obtained by the first author and the results in Sections 4 and 5.1 were obtained by the second author.

The authors wish to dedicate this paper to Hillel Furstenberg with deep mathematical and personal appreciation.

2. Basic definitions and some preliminaries

Cantor proved that the order type of $(\mathbb{Q}, <)$ is characterized up to isomorphism among all countable order types by being dense and with no end-points. Thus, any two countable dense subsets of $\mathbb{R}$ are order isomorphic to each other. As $\mathbb{Q}$ and $\pi\mathbb{Q}$ demonstrate, not all of them are isometric to each other; in fact there are $2^{\aleph_0}$ countable dense subsets of $\mathbb{R}$ no two of which are comparable with respect to isometric embedding, since the set of distances which occur in a metric space is an isometry invariant.

We introduce now relations of similarity and embeddability which are quite close to isometry and isometric embedding, but with respect to which the set of distances in a metric space is not preserved.

Definition 1:

(1) A map $f\colon X \to Y$ between metric spaces satisfies the Lipschitz condition with constant $\lambda > 0$ if for all $x_1, x_2 \in X$, $d(f(x_1), f(x_2)) < \lambda d(x_1, x_2)$.

(2) Two metric spaces X and Y are **almost isometric** if for each $\lambda > 1$ there is a homeomorphism $f\colon X \to Y$ such that f and f^{-1} satisfy the Lipschitz condition with constant λ.

(3) X is **almost isometrically embedded** in Y if for all $\lambda > 1$ there is an

injection $f: X \to Y$ so that f and f^{-1} satisfy the Lipschitz condition with constant λ.

Let us call, for simplicity, an injection f so that f and f^{-1} satisfy the λ-Lipschitz condition, **λ-bi-Lipschitz**. Observe that we use a strict inequality in the definition of the Lipschitz condition. We also note that X and Y are almost isometric if and only if the *Lipschitz distance* between X and Y is 0. The Lipschitz distance is a well-known semi-metric on metric spaces (see [7]).

It is important to notice that one does not require in (3) that the injections f for different λ have the same range.

In every infinite-dimensional Hilbert space there are two closed subsets which are almost isometric but not isometric: fix an orthonormal set of vectors $\{v_n : n \in \mathbb{N}\}$ and fix a partition $\mathbb{Q} = A_1 \cup A_2$ to two (disjoint) dense sets and enumerate $A_i = \{q^i_n : n \in \mathbb{N}\}$ for $i = 1, 2$. Let $X_i = \bigcup_{n \in \mathbb{N}} [0, q^i_n v_n]$. Now X_1, X_2 are closed (and connected) subsets of the Hilbert space which are not isometric, but are almost isometric because A_1, A_2 are.

Almost isometry and almost isometric embedding can be viewed as isomorphisms and monomorphisms of a category. Let $\mathcal{M}$ be the category in which the objects are metric spaces and the morphisms are defined as follows: a morphism from A to B is a sequence $\vec{f} = \langle f_n : n \in n \rangle$ where for each n, $f_n: A \to B$ satisfies the Lipschitz condition with a constant λ_n so that $\lim_n \lambda_n = 1$. The identity morphism $\vec{\mathrm{id}_A}$ is the constant sequence $\langle \mathrm{id}_A : n \in \mathbb{N} \rangle$ and the composition law is $\vec{g} \circ \vec{f} = \langle g_n \circ f_n : n \in \mathbb{N} \rangle$.

A morphism $\overline{f}$ in this category is invertible if and only if each f_n is invertible and its inverse satisfies the Lipschitz condition for some θ_n so that $\lim_n \theta_n = 1$, or, equivalently, $\vec{f} \in \hom(A, B)$ is an isomorphism if and only if f_n satisfies a bi-Lipschitz condition with a constant λ_n so that $\lim_n \lambda_n = 1$. Thus, A and B are isomorphic in this category if and only if for all $\lambda > 1$ there is a λ-bi-Lipschitz homeomorphism between A and B.

In Section 3 and in Section 5.1 below we shall use the following two simple facts:

FACT 2: Suppose X is a nonempty finite set, $E \subseteq X^2$ symmetric and reflexive, and (X, E) is a connected graph. Suppose that $f: E \to \mathbb{R}^+$ is a symmetric function such that $f(x, y) = 0 \iff x = y$ for all $(x, y) \in E$. Let $\sum_{i<\ell} f(x_i, x_{i+1})$ be the **length** of a path $(x_0, x_1, \ldots, x_\ell)$ in (X, E). Define $d(x, y)$ as the length of the shortest path from x to y in (X, E). Then d is a metric on X. If, furthermore, for all $(x, y) \in \mathrm{dom} f$, (x, y) is the shortest path from x to y, then d extends f. We call d as defined here the **shortest path metric**.

FACT 3: Suppose $(\{x_0,\dots,x_{n-1},x_n\},d_1)$ and $(\{y_0,\dots,y_{n-1}\},d_2)$ are metric spaces, $\theta>1$ and the mapping $x_i\mapsto y_i$ for all $i<n$ is θ-bi-Lipschitz. Then there is some $y_n\notin\{y_0,\dots,y_{n-1}\}$ and a metric d^* on $\{y_0,\dots,y_{n-1},y_n\}$ extending d_2 such that the extended mapping $x_i\mapsto y_i$ for $i\le n$ is θ-bi-Lipschitz from $(\{x_0,\dots,x_{n-1},x_n\},d_1)$ to $(\{y_0,\dots,y_{n-1}\},d^*)$

Proof: Fix $1<\lambda<\theta$ so that the mapping $x_i\mapsto y_i$ is λ-bi-Lipschitz. Fix a new point $y_n\notin\{y_0,\dots,y_{n-1}\}$ and define a function d' by letting, for each $i<n$, $d'(y_n,y_i)=d'(y_i,y_n):=\lambda d_1(x_n,x_i)$ and letting $d'(y_n,y_n)=0$. So $f=d_2\cup d'$ is a symmetric function on $\{y_0,\dots,y_{n-1},y_n\}^2$. Using Fact 2, let d^* be the shortest path metric on $\{y_0,\dots,y_{n-1},y_n\}$ obtained from $f=d_2\cup d'$.

Let us verify that d^* extends d_2. For $i,j<n$, $d_2(y_i,y_j)\le\lambda d_1(x_i,x_j)\le\lambda(d_1(x_n,x_i)+d_1(x_n,x_j))=d'(y_n,y_i)+d'(y_n,y_j)$, so (y_i,y_j) is the shortest path from y_i to y_j.

Now let us verify that the mapping $x_i\mapsto y_i$ for all $i\le n$ is θ-bi-Lipschitz. The path (y_n,y_i) has length $\lambda d_1(x_n,x_i)$ with respect to $f=d_2\cup d'$, hence the shortest path from y_n to y_i cannot be longer than $\lambda d_1(x_n,x_i)$, and hence $d^*(y_n,y_i)\le\lambda d_1(x_n,x_i)$. For the opposite inequality, fix $j<n$ so that the shortest path from y_n to y_i is (y_n,y_j,y_i). Then $d^*(y_j,y_i)\ge d_1(x_j,x_i)/\lambda$ and certainly $d^*(y_n,y_j)>d_1(x_n,x_j)$, so $d^*(y_n,y_i)\ge(d_1(x_n,x_j)+d_1(x_j,x_i))/\lambda\ge d_1(x_n,x_i)/\lambda$ as required. ∎

We conclude this section with a construction of two closed sets $A,B\subseteq\mathbb{R}$ which are not almost isometric to each other although each is almost isometrically embedded in the other.

CLAIM 4: *Suppose $X\subseteq\mathbb{R}$ is finite. Then there is some $K>1$ so that every K-bi-Lipschitz $f\colon X\to\mathbb{R}$ is either order preserving or order reversing.*

Proof: Suppose $X=\{a,b,c\}\subseteq\mathbb{R}$ and $a<b<c$. If K is such that $\max\{Kd(a,b),Kd(c,b)\}<d(a,c)/K$ and $f\colon X\to\mathbb{R}$ is K-bi-Lipschitz and, wlog, $f(a)<f(c)$, then necessarily $f(a)<f(b)<f(x)$. For a finite set X with at least 3 points let K be good enough for all 3-subsets of X. If $f\colon X\to\mathbb{R}$ is K-bi-Lipschitz and not order preserving, then there is a triple on which f is neither order preserving not order reversing, contrary to the choice of K. ∎

FACT 5: There are two closed subsets of $\mathbb{R}$ which are not almost isometric but such that each is almost isometrically embeddable into the other.

Proof: By induction on $100\le n\in\mathbb{N}$ define A_n,B_n,f_n,g_n so that:

(1) $A_{100} = \{0\}$, $B_{100} = \{0, \pi/22\}$ and f_{100}: $A_0 \to B_0$ is the identity function.
(2) A_n and B_n are finite subsets of $\{0\} \cup (1/8, 1/7)$, $A_n \cap B_0 = \{0\}$, $A_n \subseteq \mathbb{Q}$ and $B_n \cap \mathbb{Q} = \{0\}$.
(3) $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$.
(4) f_n: $A_n \to B_n$ is $(1 + 1/n)$-bi-Lipschitz and $f_n(0) = 0$.
(5) $B_{n+1} = B_n \cup \operatorname{ran} f_n$.
(6) g_n: $B_n \to A_{n+1}$ is $(1 + 1/n)$-bi-Lipschitz, $g_n(0) = 0$ and $g_n(y) \in \mathbb{Q} \setminus A_n$ for all $y \in B_n \setminus \{0\}$.
(7) $A_{n+1} = A_n \cup \operatorname{ran} g_n$.

Now set $A = \bigcup\{n + A_n : 100 \leq n \in \mathbb{N}\}$ and $B = \bigcup\{n + B_n : 100 \leq n \in \mathbb{N}\}$. So A and B have order-type ω and meet every bounded interval at most finitely. Both a and B are topologically discrete spaces, so in particular closed subsets of $\mathbb{R}$.

FACT 6: For all n, the map $x \mapsto x + n$ isometrically embeds A into A and B into B, since $n + A \cap [k, k + 1) \subseteq A \cap [k + n, k + n + 1)$ for all k.

Let $f(x) = n + f_n(x - n)$ for all $x \in A \cap [n, n + 1)$ and $g_n(y) = n + g_n(y - n)$ for all $y \in B \cap [n, n + 1)$, for each n.

Let $f^n = f{\restriction}[n, \infty)$ and let $g^n = f{\restriction}[n, \infty)$. So $f(n) = g(n) = n$ for all $100 \leq n \in \mathbb{N}$.

CLAIM 7: *For every $m \geq 100$, the function f^m is a $(1 + 1/n)$bi-Lipschitz embedding of $A \cap [m, \infty)$ into B, and the function g^m is a $(1 + 1/m)$-bi-Lipschitz embedding of $B \cap [m, \infty)$ into A.*

Proof: If $x_1, x_2 \in A \cap [k, k + 1/7)$ for some $k \geq m$, then $f{\restriction}\{x_1, x_2\}$ is $(1 + 1/k)$-bi-Lipschitz. If $x_1 \in A \cap [k, k + 1/7)$, $y \in A \cap [\ell, \ell + 1/7)$ and $m \leq k < \ell$, then $(x_1 - k)/(1 + 1/k) < f(x_1) - k < (1 + 1/k)(x_1 - k)$ and similarly for x_2 and ℓ, so $f(x_2) - f(x_1) < (x_2 - x_1) + |f(x_2) - x_2| + |f(x_1) - x_1| < x_2 - x_1$ $+2\max\{x_1 - k, x_2 - \ell\}/k < x_2 - x_1 + 1/2k < (1 + 1/k)(x_2 - x_1)$ since $x_2 - x_1 > 1/2$. ∎

For each $n \geq 100$ the function $f(x) := f^n(x + n)$ is a composition of an isometry with a $(1 + 1/n)$-bi-Lipschitz embedding and therefore is a bi-Lipschitz embedding of A into B. Thus, A is almost isometrically embeddable into B.

Similarly, B is almost isometrically embeddable into A.

It is obvious that A and B are not isometric. We argue that they are not even almost isometric. Suppose that f: $A \to B$ is a bi-Lipschitz homeomorphism with a bi-Lipschitz constant $1 < K < 8/7$. Then $f(n) \in \mathbb{N}$ for all $n \in A \cap \mathbb{N}$, and consequently, since $f \restriction \mathbb{N}$ has to be order preserving, $f \restriction (A \cap \mathbb{N})$ is the identity function. But now contradiction arises, since $A \cap [100, 101] = \{100, 101\}$ while $B \cap [100, 101] = \{100, 100 + \pi/22, 101\}$ and there is no possible preimage for $100 + \pi/22$ in A. ∎

Definition 8: Let $LAut(X)$ be the group of all autohomeomorphisms of X which are λ-bi-Lipschitz for some $\lambda > 1$. Let

$$LAut_\lambda(X) = \{f \in LAut(X) : f \text{ is } \lambda\text{-bi-Lipschitz}\}.$$

X is **almost ultrahomogeneous** if every finite λ-bi-Lipschitz map $f \colon A \to B$ between finite subsets of X extends to a λ-bi-Lipschitz autohomeomorphism.

3. Almost-isometry uniqueness and countable dense sets

Definition 9: A metric space X is **almost isometry unique** if every metric space Y which is almost isometric to X is isometric to X.

All compact metric spaces are of course almost isometry unique. In this section we shall prove that various non-compact metric spaces are almost-isometry unique and prove that the isometry types of all countable dense subsets of Uryson's space $\mathbb{U}$ form exactly one almost isometry class.

THEOREM 10: *Suppose X satisfies:*
1. *all closed bounded balls in X are compact;*
2. *there is $x_0 \in X$ and $r > 0$ so that for all $y \in X$ and all $\lambda > 1$ there is a λ-bi-Lipschitz autohomeomorphism of X so that $d(f(y), x_0) < r$.*

Then X is almost-isometry unique.

Proof: Suppose X, $x_0 \in X$ and $r > 0$ are as stated, and suppose Y is a metric space and $f_n \colon Y \to X$ is a $(1 + 1/n)$-bi-Lipschitz homeomorphism for all $n > 0$. Fix some $y_0 \in Y$. By following each f_n with a bi-Lipschitz autohomeomorphism of X (using condition (2)), we may assume that $d(f_n(y_0), x_0) < r$ for all n.

Condition (1) implies that X is separable and complete, and since Y is homeomorphic to X, Y is also separable. Fix a countable dense set $A \subseteq Y$. Since for each $a \in A$, $d(f_n(a), x_0)$ is bounded by some L_a for all n, condition (1) implies that there is a converging subsequence $\langle f_n(a) : n \in D_a \rangle$ and, since A is countable, diagonalization allows us to assume that $f_n(a)$ converges for every $a \in A$ to a point we denote by $f(a)$. The function f we defined on A is clearly an isometry, and hence can be extended to an isometry on Y. It can be verified that $f_n(y)$ converges pointwise to $f(y)$ for all $y \in Y$. Since each f_n is onto X, necessarily also f is onto X. Thus X is isometric to Y. ∎

COROLLARY 11: *For each $d \in \mathbb{N}$, $\mathbb{R}^d$ and the d-dimensional hyperbolic space $\mathbb{H}^d$ are almost-isometry unique.*

THEOREM 12 (Hrusak, Zamora-Aviles [8]): *Any two countable dense subsets of $\mathbb{R}^n$ are almost isometric. Any two countable dense subsets of the separable infinite-dimensional Hilbert space are almost isometric.*

If one regards a fixed $\mathbb{R}^d$ as a universe of metric spaces, namely considers only the subsets of $\mathbb{R}^d$, then among the countable ones there is a universal element with respect to almost isometric embedding, which is unique up to almost isometry:

COROLLARY 13: *Every dense subset of $\mathbb{R}^d$ is almost-isometry universal in the class of countable subspaces of $\mathbb{R}^d$.*

Proof: Let $B \subseteq \mathbb{R}^d$ be countable and extend B to a countable dense $A' \subseteq \mathbb{R}^d$. Since A' and A are almost isometric, B is almost isometrically embeddable into A. ∎

3.1. THE URYSON SPACE. Uryson's universal separable metric space $\mathbb{U}$ is characterized up to isometry by separability and the following property:

Definition 14 (Extension property): A metric space X satisfies the **extension property** if for every finite $F = \{x_0, \dots, x_{n-1}, x_n\}$, every isometry $f : \{x_0, \dots, x_{n-1}\} \to X$ can be extended to an isometry $\hat{f}: F \to X$.

Separability together with the extension property easily imply the isometric uniqueness of $\mathbb{U}$ as well as the fact that every separable metric space is isometric to a subspace of $\mathbb{U}$ and that $\mathbb{U}$ is ultrahomogeneous, namely, every isometry between finite subspaces of $\mathbb{U}$ extends to an auto-isometry of $\mathbb{U}$. This property of $\mathbb{U}$ was recently used to determine the Borel complexity of the isometry relation on polish spaces [3, 6].

Definition 15 (Almost extension property): A metric space X satisfies the **almost extension property** if for every finite space $F = \{x_0, \dots, x_{n-1}, x_n\}$ and $\lambda > 1$, every λ-bi-Lipschitz $f: \{x_0, \dots, x_{n-1}\} \to X$ can be extended to a λ-bi-Lipschitz $\hat{f}: F \to X$.

CLAIM 16: *Suppose that $A \subseteq \mathbb{U}$ is dense in $\mathbb{U}$. Then A satisfies the almost extension property.*

Proof: Suppose $f\colon \{x_1, \dots, x_{n-1}\} \to A$ is λ-bi-Lipschitz. By Lemma 3 there is a metric extension $\operatorname{ran} f \cup \{y_n\}$ so that $f \cup \{(x_n, y_n)\}$ is λ-bi-Lipschitz. By the extension property of $\mathbb{U}$, we may assume that $y_n \in \mathbb{U}$. Now replace y_n by a sufficiently close $y'_n \in A$ so that $f \cup \{(x_n, y'_n)\}$ is λ-bi-Lipschitz. ∎

A standard back and forth argument shows:

FACT 17: Any two countable metric spaces that satisfy the almost extension property are almost isometric.

Therefore we have proved:

THEOREM 18: *Any two countable dense subsets of $\mathbb{U}$ are almost isometric.*

A **type** p over a metric space X is a function $p\colon X \to \mathbb{R}^+$ so that in some metric extension $X \cup \{y\}$, $d(y,x) = p(x)$ for all $x \in X$. A point $y \in Y$ **realizes** a type p over a subset $X \subseteq Y$ if $p(x) = d(y,x)$ for all $x \in X$. The extension property of $\mathbb{U}$ is equivalent to the property that every type over a finite subset of $\mathbb{U}$ is realized in $\mathbb{U}$.

THEOREM 19 (Uryson): *If a countable metric space A satisfies the almost extension property, then its completion $\bar{A}$ satisfies the extension property and is therefore isometric to $\mathbb{U}$.*

Proof: Let $X \subseteq \bar{A}$ be a finite subset and let p be a metric type over X. Given $\varepsilon > 0$, find $\lambda > 1$ sufficiently close to 1 so that for all $x \in X$, $\lambda p(x) - p(x) < \varepsilon/2$ and $p(x) - p(x)/\lambda < \varepsilon/2$, and find, for each $x \in X$, some $x' \in A$ with $d(x', x) < \varepsilon/2$ and sufficiently small so that the map $x \mapsto x'$ is λ-bi-Lipschitz. By the almost extension property of A there is some $y \in A$ so that $d(y,x) < \varepsilon$. Thus we have shown that for all finite $X \subseteq \bar{A}$ and type p over X, for every $\varepsilon > 0$ there is some point $y \in A$ so that $|d(y,x) - p(x)| < \varepsilon$ for all $x \in X$.

Suppose now that $X \subseteq \bar{A}$ is finite and that p is some type over X. Suppose $\varepsilon > 0$ is small, and that $y \in A$ satisfies $|d(y,x) - p(x)| < \varepsilon$ for all $x \in X$. Extend the type p to $X \cup y$ by putting $p(y) = 2\varepsilon$ (since $|d(y,x) - p(x)| < \varepsilon$, this is indeed a type). Using the previous fact, find $z \in A$ that realizes p up to $\varepsilon/100$ and satisfies $d(y,z) < 2\varepsilon$.

Iterating the previous paragraph, one gets a Cauchy sequence $(y_n)_n \subseteq A$ so that for all $x \in X$, $d(y_n, x) \to p(x)$. The limit of the sequence satisfies p in $\bar{A}$. ∎

We now have:

FACT 20: A countable metric space is isometric to a dense subset of $\mathbb{U}$ if and only if it satisfies the almost extension property. Thus, the isometry-types of all countable dense subsets of $\mathbb{U}$ form exactly one almost-isometry equivalence class.

THEOREM 21: *The Uryson space $\mathbb{U}$ is almost isometry unique.*

Proof: Suppose X is almost isometric to $\mathbb{U}$. Fix a countable dense $A \subseteq \mathbb{U}$ and a countable dense $B \subseteq X$. For every $\lambda > 1$, B is λ-bi-Lipschitz homeomorphic to *some* countable dense subset of $\mathbb{U}$, so by Theorem 18 it is λ^2-bi-Lipschitz homeomorphic to A. So A and B are almost isometric. This shows that B has the almost extension property. By Uryson's theorem, $\bar{B} = Y$ is isometric to $\mathbb{U}$. ∎

Let us construct now, for completeness of presentation, *some* countable dense subset of $\mathbb{U}$.

Let $\langle A_n : n \in \mathbb{N} \rangle$ be an increasing sequence of finite metric spaces so that:

(1) all distances in A_n are rational numbers;
(2) for every function p: $A_n \to \mathbb{Q}^n$ which satisfies the triangle inequality $p(x_1)+d(x_1,x_2) \geq p(x_2)$ and $p(x_1)+p(x_2) \geq d(x_1,x_2)$ for all $x_1, x_2 \in A_n$, *and* satisfies that $p(x) \leq n+1$ is a rational number with denominator $\leq n+1$, there is $y \in A_{n+1}$ so that $p(x) = d(y,x)$ for all $x \in A_n$.

Such a sequence obviously exists. A_0 can be taken as a singleton. To obtain A_{n+1} from A_n one adds, for each of the finitely many distance functions p as above, a new point that realizes p, and then sets the distance $d(y_1, y_1)$ between two new points to be $\min\{d(y_1,x) + d(y_2,x) : x \in A_n\}$. Let $\mathbb{A} := \bigcup_n A_n$. To see that $\mathbb{A}$ satisfies the almost extension property, one only needs to verify that every type p over a finite rational X can be arbitrarily approximated by a rational type (we leave that to the reader).

This construction, also due to Uryson, shows that the Uryson space has a dense rational subspace. (Another construction of $\mathbb{U}$ can be found in [7].) This is a natural place to recall:

PROBLEM 22 (Erdős): *Is there a dense rational subspace of $\mathbb{R}^2$?*

PROBLEM 23: *Is it true that for a separable and homogeneous complete metric space, all countable dense subsets are almost isometric if and only if the space is almost-isometry unique?*

4. Almost isometric embeddability between $\aleph_1$-dense sets

Definition 24: A subset A of a metric space X is **$\aleph_1$-dense** if for every nonempty open ball $u \subseteq X$, $|A \cap u| = \aleph_1$.

Among the early achievements of the technique of forcing, a place of honor is occupied by Baumgartner's proof of the consistency of "all $\aleph_1$-dense sets of reals are order isomorphic" [1]. Early on, Sierpinski proved that there are $2^{2^{\aleph_0}}$ order-incomparable continuum-dense subsets of $\mathbb{R}$, hence Baumgartner's consistency result necessitates the failure of CH.

Today it is known that Baumgartner's result follows from forcing axioms like PFA and also from Woodin's axiom $(*)$ [19].

In our context, one can inquire about the consistency of two natural analogs of the statement whose consistency was established by Baumgartner. First, is it consistent that all $\aleph_1$-dense subsets of $\mathbb{R}$ are almost isometric? Since every bi-Lipschitz homeomorphism between two dense subsets of $\mathbb{R}$ is either order preserving or order inverting, and since there is an $\aleph_1$-dense subset of $\mathbb{R}$ which is order isomorphic to its reflection on 0, this statement is stronger than Baumgartner's consistent statement. Second, since $\mathbb{U}$ in the category of metric spaces has the role $\mathbb{R}$ has in the category of separable linear orders (it is the universal separable object), is it consistent that all $\aleph_1$-dense subsets of $\mathbb{U}$ are almost isometric?

The answer to both questions is negative.

4.1. Incomparable $\aleph_1$-dense sets. For every infinite $A \subseteq 2^{\mathbb{N}}$, let $T_A \subseteq 2^{<\mathbb{N}}$ be defined (inductively) as the tree T which contains the empty sequence and for every $\eta \in T$ contains $\eta\frown 0, \eta\frown 1$ if $|\eta| \in A$, and contains only $\eta\frown 0$ if $|\eta| \notin A$. Let D_A be the set of all infinite branches through T_A. The set of positive distances occurring in D_A, namely $\{d(x,y) : x, y \in D_A, x \neq y\}$, is equal to $\{1/2^n : n \in A\}$.

Let d_3 be the metric on $2^{\mathbb{N}}$ defined by $d_3(\eta,\nu) := 1/3^{\Delta(\eta,\nu)}$. Observe that the natural isomorphism between $2^{\mathbb{N}}$ and the standard "middle-third" Cantor set is a 4-bi-Lipschitz map when $2^{\mathbb{N}}$ is taken with d_3.

For sets $A, B \subseteq N$ let us define the following condition:

$(*)$ $|A| = |B| = \aleph_0$ *and for every n there is k so that for all $a \in A, b \in B$, if $a, b > k$ then $a/b > n$ or $b/a > n$.*

Lemma 25: *Suppose $A, B \subseteq \mathbb{N}$ satisfy $(*)$. Then for every infinite set $X \subseteq D_A$ and every function $f\colon X \to D_B$, for every $n > 1$ there are distinct $x, y \in X$ so that either $d(f(x), f(y))/d(x,y) > n$ or $d(f(x), f(y))/d(x,y) < 1/n$.*

Proof: Let $x, y \in X$ be chosen so that $d(x, y) > 0$ is sufficiently small. ∎

The lemma assures a strong form of bi-Lipschitz incomparability: no infinite subset of one of the spaces D_A, D_B can be bi-Lipschitz embeddable into the other space, if A, B satisfy $(*)$.

Let $\mathcal{F}$ be an almost disjoint family over $\mathbb{N}$, namely, a family of infinite subsets of $\mathbb{N}$ with finite pairwise intersections. Replacing each $A \in \mathcal{F}$ by $\{n^n : n \in A\}$, one obtains a family of sets that pairwise satisfy condition $(*)$. Since there is an almost disjoint family of size $2^{\aleph_0}$ over $\mathbb{N}$, there is a family $\mathcal{F}$ of subsets of $\mathbb{N}$ whose members satisfy $(*)$ pairwise, and therefore $\{D_A : A \in \mathcal{F}\}$ is a family of pairwise bi-Lipschitz incomparable perfect subspaces of $(2^{\mathbb{N}}, d_3)$ of size $2^{\aleph_0}$.

THEOREM 26: *Suppose X is a separable metric space and that for some $K \geq 1$, for every open ball u in X there is a (nonempty) open subset of $(2^{\mathbb{N}}, d_3)$ which is K-bi-Lipschitz embeddable into u. Then there are $2^{\aleph_0}$ pairwise bi-Lipschitz incomparable $\aleph_1$-dense subsets of X.*

In particular, in every separable Hilbert space (finite- or infinite-dimensional) and in $\mathbb{U}$ there are $2^{\aleph_0}$ pairwise bi-Lipschitz incomparable $\aleph_1$-dense subsets.

Proof: Fix a family $\{D_\alpha : \alpha < 2^{\aleph_0}\}$ of pairwise bi-Lipschitz incomparable perfect subspaces of $2^{\mathbb{N}}$, as above. For each α, fix an $\aleph_1$-dense $D^*_\alpha \subseteq D_\alpha$.

Let $\langle u_n : n \in \mathbb{N}\rangle$ enumerate a basis for the topology of X (say all balls of rational radii with centers in some fixed countable dense set). For each $\alpha < 2^{\aleph_0}$, for each n, since there is a 4-bi-Lipschitz embedding of *some* nonempty basic open set of $(2^{\mathbb{N}}, d_3)$ into u_n, there is also a 4-bi-Lipschitz embedding of a nonempty open subset of D^*_α into u_n. Fix such an embedding and call the image of the embedding $E^*_{\alpha,n}$. Let $Y_\alpha = \bigcup_n E^*\alpha, n$. Y_α is thus an $\aleph_1$-dense subset of X.

Suppose f: $Y_\alpha \to Y_\beta$ is any function, $\alpha, \beta < 2^{\aleph_0}$ distinct, and $K > 1$ is arbitrary. There is $l \in \mathbb{N}$ so that $f^{-1}[E^*_{\beta,l}] \cap E^*_{\alpha,0}$ is uncountable, hence infinite. Therefore, by Lemma 25, there are $x, y \in Y_\alpha$ so that $d(f(x), f(y))/d(x, y)$ violates n-bi-Lipschitz. We conclude that Y_α, Y_β are bi-Lipschitz incomparable.

The second part of the theorem follows now from the fact that every separable metric space embeds isometrically into $\mathbb{U}$ and from the observation above that $(2^{\mathbb{N}}, d_3)$ has a bi-Lipschitz embedding into $\mathbb{R}$. ∎

5. Universality of metric spaces below the continuum

Let $(\mathcal{M}^{sep}_{\aleph_1}, \leq)$ denote the set of all (isometry types of) separable metric spaces whose cardinality is $\aleph_1$, quasi-ordered by almost isometric embeddability, and let $(\mathcal{M}_{\aleph_1}, \leq)$ denote the set of all (isometry types of) metric spaces whose cardinality is $\aleph_1$, quasi-ordered similarly.

Let $\mathrm{cf}(\mathcal{M}_{\aleph_1}, \leq)$ denote the **cofinality** of this quasi-ordered set: the least cardinality of $D \subseteq \mathcal{M}^{sep}_{\aleph_1}$ with the property that for every $M \in \mathcal{M}^{sep}_{\aleph_1}$ there is $N \in D$ so that $M \leq N$. The statement "$\mathrm{cf}(\mathcal{M}^{sep}_{\aleph_1}, \leq) = 1$" means that there is a single $\aleph_1$-dense subset of $\mathbb{U}$ in which every $\aleph_1$-dense subset of $\mathbb{U}$ is almost isometrically embedded, or, equivalently, that there is a *universal* separable metric space of size $\aleph_1$ for almost isometric embedding.

In the previous section it was shown that there are $2^{\aleph_0}$ pairwise incomparable elements — an anti-chain — in this quasi-ordering, with each of the elements being an $\aleph_1$-dense subset of $\mathbb{U}$. This in itself does not rule out the possibility that a universal separable metric space of size $\aleph_1$ exists for almost isometric embedding. In fact, if CH holds, $\mathbb{U}$ itself is such a set.

What can one expect if CH fails? There has been a fairly extensive study of the problem of universality below $2^{\aleph_0}$ in various classes of structures [14, 15, 16, 10, 11, 12, 13, 4, 5]. It is fairly routine to produce models of set theory in which there are no universal objects in cardinality $\aleph_1$ in every reasonable class of structures (linear orders, graphs, etc.) [13, 10] and here too it is routine to find models in which neither a universal separable metric space nor a universal general metric space exists in cardinality $\aleph_1$ (Fact 28 below).

It has been shown that universal linear orderings may consistently exist in $\aleph_1 < 2^{\aleph_0}$ [14], that universal graphs may consistently exist in any prescribed regular $\lambda < 2^{\aleph_0}$ [15, 16] and, more generally, that universal relational theories with certain amalgamation properties may consistently have universal models in regular uncountable $\lambda < 2^{\aleph_0}$ [13]. The amalgamation properties which the class of graphs satisfies are not available in the class of linear orderings[1], and, in strong contrast to graphs, universal linear orderings cannot exist in a regular cardinal $\lambda > \aleph_1$ below the continuum [10].

In the class of metric spaces with almost isometric embedding, the situation is as follows. Metric spaces do not satisfy 3-amalgamation, and indeed behave

1 A class of structures satisfies the n-amalgamation property if, whenever M_X is a structure from the class for each $X \in \mathcal{P}(n) \setminus \{n\}$ so that $X_1 \subseteq X_2 \Rightarrow M_{X_1} \leq M_{X_2}$, there is a structure M_n so that $M_X \leq M_n$ for all $X \in \mathcal{P}(n)$. The class of graphs satisfies n-amalgamation for all $n > 1$. Linear orderings satisfy 2-amalgamation but not 3-amalgamation. Metric spaces also satisfy 2- but not 3-amalgamation (let $M_{\{0,1\}} = \{a, b\}$ with $d(a, b) = 1$, $M_{\{1,2\}} = \{b, c\}$ with $d(b, c) = 3$ and $M_{\{0,2\}} = \{a, c\}$ with $d(a, c) = 1$. Then the union of any two of the three metrics can be extended to a metric, but the union of all three metrics violates the triangle inequality).

like linear orderings: there cannot exist a universal metric space with respect to almost isometric embedding in any regular $\lambda > \aleph_1$ below the continuum (Theorem 35 below). However, the additional assumption of separability partially compensates for the absence of amalgamation, and it is consistent to have a small number of separable metric spaces on a prescribed regular $\lambda < \aleph_0$ below the continuum so that every separable metric space on λ is almost isometrically embeddable into one of them (Theorem 34 below).

We do not know whether it is consistent to have a universal separable metric space of size $\aleph_1 < 2^{\aleph_0}$ for almost isometric embedding.

We begin by relating separable to non-separable spaces:

THEOREM 27: $\mathrm{cf}(\mathcal{M}^{sep}_{\leq\aleph_n}, \leq) \leq \mathrm{cf}(\mathcal{M}_{\aleph_n}, \leq)$ *for all* n.

Proof: Suppose $M = (\omega_1, d)$ is a metric space. For every ordinal $\alpha < \omega_1$ denote the closure of α in (M, d) by X_α. The space X_α is separable and is therefore isometric to some $Y_\alpha \subseteq \mathbb{U}$. Let $N(M) = \bigcup_{\alpha<\omega_1} Y_\alpha$. $N(M)$ is a subspace of $\mathbb{U}$ whose cardinality is $\leq \aleph_1$.

Suppose X is an arbitrary separable subspace of M, and fix a countable dense $A \subseteq X$. There is some $\alpha < \omega_1$ so that $A \subseteq \alpha$, hence $X \subseteq X_\alpha$ and is thus isometrically embedded into $Y_\alpha \subseteq N$. In other words, $N(M)$ is a single subspace of $\mathbb{U}$ of cardinality $\leq \aleph_1$ into which all *separable* subspaces of M are isometrically embedded.

Suppose now that $\mathrm{cf}(\mathcal{M}_{\aleph_1}, \leq) = \kappa$ and fix $D \subseteq \mathcal{M}_{\aleph_1}$ of cardinality κ so that for all $N \in \mathcal{M}_{\aleph_1}$ there is $M \in D$ so that $N \leq D$. For each $M \in D$ let $N(M) \subseteq \mathbb{U}$ be chosen as above. We claim that $\{N(M) : M \in D\}$ demonstrates that $\mathrm{cf}(\mathcal{M}^{sep}_{\leq\aleph_1}, \leq) \leq \kappa$. Suppose that X is a separable metric space of cardinality $\aleph_1$. Then X is almost isometrically embedded into M for some $M \in D$. Since every separable subspace of M is isometric to a subspace of $N(M)$, it follows that X is almost isometrically embedded in $N(M)$.

Simple induction on n shows that for every n there is a collection $\mathcal{F}_n$ of $\aleph_n$ many countable subsets of ω_n with the property that every countable subset of ω_n is contained in one of them. Working with $\mathcal{F}_n$ instead of the collection of initial segments of ω_1 gives that $\mathrm{cf}(\mathcal{M}^{sep}_{\leq\aleph_n}, \leq) \leq \mathrm{cf}(\mathcal{M}_{\aleph_n}, \leq)$. ∎

The next fact is standard, and we include its proof only for completeness of presentation.

FACT 28: After adding $\lambda \geq \aleph_2$ Cohen reals to a universe V of set theory, $\mathrm{cf}(\mathcal{M}^{sep}_{\aleph_1}, \leq) \geq \lambda$.

Proof: View adding λ Cohen reals as an iteration. Let $\theta < \lambda$ be given. For any family $\{A_\alpha : \alpha < \theta\}$ of $\aleph_1$-dense subsets of $\mathbb{U}$ in the extension it may be assumed, by using θ of the Cohen reals, that $A_\alpha \in V$ for all $\alpha < \theta$. Let $X = \mathbb{Q} \cup \{r_i : i < \omega_1\}$ be a metric subspace of $\mathbb{R}$, where each r_i, for $i < \omega_1$, is one of the Cohen reals. We argue that X cannot be almost isometrically embedded into any A_α. Suppose to the contrary that $f\colon X \to A_\alpha$ is a bi-Lipschitz embedding. By using countably many of the Cohen reals, we may assume that $f\restriction\mathbb{Q} \in V$. If $f(r_0) \in A_\alpha$ and $f\restriction\mathbb{Q}$ are both in V, so is r_0 — contradiction. ∎

5.1. Consistency for separable spaces. For the next consistency result we need the following classic result of Baumgartner:

Theorem 29 ([2]): *For every regular $\lambda > \aleph_0$ and regular $\theta > \lambda^+$ there is a model V of set theory in which $2^\kappa = \theta$ for every $\aleph_0 \le \kappa < \theta$ and there is a family $\{A_\alpha : \alpha < \theta\}$ of subsets of λ, each A_α of cardinality λ and $|A_\alpha \cap A_\beta| < \aleph_0$ for all $\alpha < \beta < \theta$.*

We now state and prove the consistency of having a small family of separable metric spaces on λ which together almost isometrically embed every separable metric space on λ, when $\lambda < 2^{\aleph_0}$. We first prove it for $\lambda = \aleph_1$, for simplicity. Then we extend it to a general regular $\lambda > \aleph_0$.

Theorem 30: *It is consistent that $2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$ and that there are $\aleph_2$ separable metric spaces on ω_1 such that every separable metric space is almost isometrically embedded into one of them.*

The model of set theory which demonstrates this consistency is obtained as a forcing extension of a ground model which satisfies $2^{\aleph_0} = 2^{\aleph_1} = 2^{\aleph_2} = \aleph_3$ and there are $\aleph_3$ $\aleph_1$-subsets of $\aleph_1$ with finite pairwise intersections. Such a ground model exists by Baumgartner's Theorem 29. Then the forcing extension is obtained via a ccc finite support iteration of length $\aleph_2$. In each step $\zeta < \omega_2$ a single new separable metric space M_ζ of cardinality $\aleph_1$ is forced together with almost isometric embeddings of all $\aleph_1$-dense subsets of $\mathbb{U}$ that V_ζ knows. At the end of the iteration, every $\aleph_1$-dense subset is almost isometrically embedded into one of the spaces M_ζ that were forced.

Let $\{A_\alpha : \alpha < \omega_3\}$ be a collection of subsets of ω_1, each of cardinality $\aleph_1$, so that for all $\alpha < \beta < \omega_3$, $A_\alpha \cap A_\beta$ is finite. For each $\alpha < \omega_3$ fix a partition $A_\alpha = \bigcup_{i<\omega_1} A_{\alpha,i}$ to $\aleph_1$ parts, each of cardinality $\aleph_0$.

Fix an enumeration $\langle d_\alpha : \alpha < \omega_3\rangle$ of all metrics d on ω_1 with respect to which (ω_1, d) is a separable metric space and every interval $(\alpha, \alpha + \omega)$ is dense in it.

Since every separable metric space of cardinality ω_1 can be well ordered in order type ω_1 so that every interval $(\alpha, \alpha+\omega)$ is dense in the whole space, this list contains ω_3 isometric copies of every separable metric space of cardinality $\aleph_1$.

We define now the forcing notion Q. A condition $p \in Q$ is an ordered quintuple $p = \langle w^p, u^p, d^p, \overline{f}^p, \overline{\varepsilon}^p \rangle$ where:

(1) w^p is a finite subset of ω_3 (intuitively — the set of metric spaces (ω_1, d_α) which the condition handles).

(2) $u^p \subseteq \omega_1$ is finite and d^p is a metric over u^p. (u^p, d^p) is a finite approximation to the space $M = (\omega_1, d)$ which Q introduces.

(3) $\overline{\varepsilon}^p = \langle \varepsilon^p_\alpha : \alpha \in w^p \rangle$ is a sequence of rational numbers from $(0,1)$.

(4) $\overline{f} = \langle f_\alpha : \alpha \in w^p \rangle$ is a sequence of finite functions $f_\alpha : (\omega_1, d_\alpha) \to (u^p, d^p)$ that satisfy:

 (a) $f^p_\alpha(i) \in A_{\alpha,i}$ for each $i \in \mathrm{dom} f^p_\alpha$;

 (b) each f^p_α is $(1+\varepsilon^p_\alpha)$-bi-Lipschitz.

The order relation is: $p \leq q$ (q extends p) iff $w^p \subseteq w^q$, (u^p, d^p) is a subspace of (u^q, d^q), and for all $\alpha \in w^p$, $\varepsilon^p_\alpha = \varepsilon^q_\alpha$ and $f^p_\alpha \subseteq f^q_\alpha$.

Informally, a condition p provides finite approximations of $(1+\varepsilon^p_\alpha)$-bi-Lipschitz embeddings of (ω_1, d_α) for finitely many $\alpha < \omega_3$ into a finite space (u^p, d^p) which approximates (ω_1, d).

Lemma 31 (Density): *For every $p \in Q$:*

(1) *For every $j \in \omega_1 \setminus u^p$ and a metric type t over (u^p, d^p) there is a condition $q \geq p$ so that $j \in u^q$ and j realizes t over u^p in (u^q, d^q).*

(2) *For every $\alpha \in \omega_3 \setminus w^p$ and $\delta > 0$ there is a condition $q \geq p$ so that $\alpha \in w^q$ and $\varepsilon^p_\alpha < \delta$.*

(3) *For every $\alpha \in w^p$ and $i \in \omega_1 \setminus \mathrm{dom} f^p_\alpha$ there is a condition $q \geq p$ so that $i \in \mathrm{dom} f^q_\alpha$*

Proof: To prove (1) simply extend (u^p, d^p) to a metric space (u^q, d^q) which contains j and in which j realizes t over u^p, leaving everything else in p unchanged.

For (2) define $w^q = w^p \cup \{\alpha\}$ and $\varepsilon^p_\alpha = \varepsilon$ for a rational $\varepsilon < \delta$.

For (3) suppose $i \notin \mathrm{dom} f^p_\alpha$. Fix some $x \in A_{\alpha,i} \setminus u^p$ and fix some $1 < \lambda < 1+\varepsilon^p_\alpha$ so that $d_\alpha(j,k)/\lambda \leq d^p(j,k) \leq \lambda d_\alpha(j,k)$ for all distinct $j, k \in \mathrm{dom} f^p_\alpha$. Let $d^*(x, f^p_\alpha(j)) = r_j$ for some rational $\lambda d_\alpha(i,j) \leq r_j < (1+\varepsilon^p_\alpha) d_\alpha(i,j)$, and let $d^*(x,x) = 0$. Let d be the shortest path metric obtained from $d^p \cup d^*$. This is obviously a rational metric and, as in the proof of Fact 3, it follows that this metric extends d^p and that $f^p_\alpha \cup \{(i,x)\}$ is $(1+\varepsilon^p_t)$-bi-Lipschitz into $u^p \cup \{x\}$ with the extended metric. ∎

LEMMA 32: *In V^Q there is a separable metric space $M = (\omega_1, d)$ so that for every separable metric space $(\omega_1, d') \in V$ and $\delta > 0$ there is a $(1+\delta)$-bi-Lipschitz embedding in V^Q of (ω_1, d') into M.*

Proof: Let $d = \bigcup_{p \in G} d^p$ where G is a V-generic filter of Q. By (1) in the density lemma, d is a metric on ω_1. Furthermore, a standard (forcing) density argument shows that for every given $i < \omega_1$ the interval $(i, i + \omega)$ is dense in (ω_1, d).

Given any metric space (ω_1, d') and a condition p, find some $\alpha \in \omega_3 \setminus w^p$ so that (ω_1, d_α) is isometric to (ω_1, d') and apply (2) to find a stronger condition q so that $\alpha \in w^q$ and $\varepsilon^q < \delta$. For every $i < \omega_1$ there is a stronger condition q' so that $i \in \mathrm{dom} f_\alpha^{q'}$. Thus, the set of conditions which force a partial $(1 + \delta)$-bi-Lipschitz embedding from (ω_1, d_α) into (ω_1, d) which includes a prescribed $i < \omega_1$ in its domain is dense; therefore Q forces a $(1+\delta)$-bi-Lipschitz embedding of (ω_1, d') into (ω_1, d). ∎

LEMMA 33: *Every antichain in Q is countable.*

This Lemma is the main ingredient in the proof of Theorem 30. Before plunging into the details of the proof, let us sketch shortly the main point in the Lemma's proof, and in particular clarify where it is different from the parallel proofs in Shelah's consistency of a universal graphs in $\lambda < 2^{\aleph_0}$. In Shelah's proofs [15, 16] the ccc is guaranteed by the amalgamation properties of the class of graphs (a fact that was abstracted from Shelah's proof by Mekler and used in [13] to generalize the results from graphs to classes of universal relational structures with amalgamation).

The required amalgamation properties are not available in the class of metric spaces. However, the topological assumption of separability partially compensates for this. While combining two arbitrary finite λ-bi-Lipschitz embeddings from a fixed metric space into a single one is not generally possible, when the domains are sufficiently "close" to each other, it is possible. The main point in the proof of the ccc condition is that separability of the spaces $(\omega_1, \delta_\alpha)$ for $\alpha < \omega_3$ implies that among any $\aleph_1$ finite disjoint subsets of (ω_1, d_α) there are two which are arbitrarily small perturbations of each other.

Proof of Lemma 33: Suppose $\{p_\zeta : \alpha < \omega_1\} \subseteq Q$ is a set of conditions. Applying the Δ-system lemma and the pigeon hole principle a few times, we may assume (after replacing the set of conditions by a subset and re-enumerating) that:

(1) $|u^{p_\zeta}| = n$ for all $\zeta < \omega_1$ for some fixed n and $\{u_\zeta : \zeta < \omega_1\}$ is a Δ-system with root u, $|u| = m$. Denote $u^{p_\zeta} = u \cup u_\zeta$ (so $\zeta < \xi < \omega_1 \Rightarrow u_\zeta \cap u_\xi = \emptyset$).
(2) $d^{p_\zeta} \restriction u = d$ for some fixed (rational) metric d.
(3) Denote by $g_{\zeta,\xi}$ the order preserving map from u_ζ onto u_ξ. Then $\mathrm{id}_u \cup g_{\zeta,\xi}$ is an isometry between u^{p_ζ} and u^{p_ξ}. This means that u_ζ and u_ξ are isometric and that for every $x \in u_\zeta$ and $y \in u$, $d^{p_\zeta}(x,y) = d^{p_\xi}(g_{\zeta,\xi}(x), y)$.
(4) $\{w^{p_\zeta} : \zeta < \omega_1\}$ is a Δ-system with root w.
(5) $\varepsilon_\alpha^{p_\zeta} = \varepsilon_\alpha$ for some fixed ε_α for every $\alpha \in w$ and $\zeta < \omega_1$.
(6) For every $\alpha \in w$, $\{\mathrm{dom} f_\alpha^{p_\zeta} : \zeta < \omega_1\}$ is a Δ-system with root r_α and $|\mathrm{dom} f_\alpha^{p_\zeta}|$ is fixed.
(7) $f_\alpha^{p_\zeta} \restriction r_\alpha$ is fixed (may be assumed since $f_\alpha^{p_\zeta}(i) \in A_{\alpha,i}$ for all $i \in r$ and $A_{\alpha,i}$ is countable).
(8) Denote $\mathrm{dom} f_\alpha^{p_\zeta} = r_\alpha \cup s_\alpha^\zeta$ for $\alpha \in w$; then $f_\alpha^{p_\zeta}[s_\alpha^\zeta] \cap u = \emptyset$. Consequently, $f_\alpha^\zeta[s_\alpha^\zeta] \subseteq u^\zeta$.
(9) For all $\alpha \in w$ and $\zeta, \xi < \omega_1$, $g_{\zeta,\xi}[f_\alpha^{p_\zeta}[s_\alpha^\zeta]] = f_\alpha^{p_\xi}[s_\alpha^\xi]$. Denote $h_\alpha^{\zeta,\xi} = (f_\alpha^{p_\xi})^{-1} \circ g_{\zeta,\xi} \circ f_\alpha^{p_\zeta}$. Thus $h_\alpha^{\zeta,\xi} \colon s_\alpha^\zeta \to s_\alpha^\xi$ and $f_\alpha^{p_\xi}(h_\alpha^{\zeta,\xi}(i)) = f_\alpha^{p_\zeta}(i)$ for all $i \in s_\alpha^\zeta$.
(10) For all $\alpha, \beta \in w$ and $\zeta < \omega_1$, $\mathrm{ran}(f_\alpha^\zeta) \cap \mathrm{ran}(f_\beta^\zeta) \subseteq u$; here we use the fact that $\mathrm{ran} f_\alpha^\zeta \subseteq A_\alpha, \mathrm{ran}(f_\beta^\zeta) \subseteq A_\beta$ and $|A_\alpha \cap A_\beta| < \aleph_0$.

Fix $\alpha \in \omega$ and a point $x \in f_\alpha^\zeta[s_\alpha^\zeta]$ (the set does not depend on ζ). Consider the set $\{i \in s_\alpha^\zeta : \zeta < \omega_1 \text{ and } f_\alpha^\zeta(i) = x\}$. This is an uncountable subset of the separable metric space (ω_1, d_α), so after thinning-out we may assume that every point in this set is a complete accumulation point of the set, that, for every $\varepsilon > 0$, $\zeta < \omega_1$ and $i \in S_\alpha^\zeta$, there are ω_1 many $\xi < \omega_1$ so that $d_\alpha(i, h_\alpha^{\zeta,\xi}(i)) < \varepsilon$.

Repeating this thinning-out a finite number of times, we may assume:

(11) For every $\alpha \in w$ and $\zeta < \omega_1$, every $i \in s_\alpha^\zeta$ is a point of complete accumulation in (ω_1, d_α) of $\{h_\alpha^{\zeta,\xi}(i) : \xi < \omega_1\}$.

We shall find now two conditions p_ζ, p_ξ, $\zeta < \xi < \omega_1$ and a condition $t \in Q$ which extends both p_ζ and p_ξ.

Fix some $\zeta < \omega_1$ and define $\delta_0 = \min\{d^{p_\zeta}(x,y) : x \neq y \in u \cup u_\zeta\}$. Next, for $\alpha \in w$ let $i, j \in \mathrm{dom} f_\alpha^{p_\zeta}$ be any pair of points. We have

$$(i) \qquad d_\alpha(i,j)/(1+\varepsilon_\zeta^{p_\zeta}) < d^{p_\zeta}(f_\zeta^{p_\zeta}(i), f_\zeta^{p_\zeta}(j)) < (1+\varepsilon_\zeta^{p_\zeta}) d_\alpha(i,j).$$

Denoting, for simplicity, $\lambda = (1+\varepsilon_\alpha^{p_\zeta})$, $a = d_\alpha(i,j)$ and $b = d^{p_\zeta}(f_\alpha^{p_\zeta}(i), f_\zeta^{p_\zeta}(j))$, this can be re-written as

$$(ii) \qquad a/\lambda < b < \lambda a.$$

There is a sufficiently small $\delta > 0$, depending on a and b, such that for all a', b', if $|b - b'| < \delta$ and $|a - a'| < \delta$ then

$$(\textit{iii}) \qquad a'/\lambda < b' < \lambda a'.$$

Since w and $\mathrm{dom} f_\alpha^{p_\zeta}$ for each $\alpha \in w$ are finite, we can fix $\delta_1 > 0$ which is sufficiently small to make (*iii*) hold for all $\alpha \in w$ and $i, j \in \mathrm{dom} f_\alpha^{p_\zeta}$.

Let $\delta = \min\{\delta_0/100, \delta_1/2\}$.

By condition (11) above, find $\zeta < \xi < \omega_1$ so that for each $\alpha \in w$ and $i \in s_\alpha^\zeta$, $d_\alpha(i, h_\alpha^{\zeta,\xi}(i)) < \delta$.

Let $u^t = u \cup u_\zeta \cup u_\xi$. We define now a metric d^t on u^t as follows. First, for each $\alpha \in w$ and $x = f_\alpha^{p_\zeta}(i) \in f_\alpha^{p_\zeta}[s_\alpha^\zeta]$ let $y = g_{\zeta,\xi}(x)$ and define $d^*(x, y) = r$, r a rational number which satisfies $1/(1 + \varepsilon_\alpha^p) d_\alpha(i, j) < r < (1 + \varepsilon_\alpha^p) d_\alpha(i, j)$, where $j = h_\alpha^{\zeta,\xi}(i)$. Since for $\alpha, \beta \in w$ the sets $f_\alpha^{p_\zeta}[s_\alpha^\zeta]$ and $f_\alpha^{p_\zeta}[s_\beta^\xi]$ are disjoint, and similarly for ξ, d^* is well defined, namely, at most one α is involved in defining the distance r. In fact, any two d^* edges are vertex disjoint.

Now $(u \cup u_\zeta \cup u_\xi, d \cup d^*)$ is a connected weighted graph. Let d^t be the shortest-path metric on $u \cup u_\zeta \cup u_\xi$ obtained from $d^{p_\zeta} \cup d^{p_\xi} \cup d^*$. It is obviously a rational metric, as all distances in $d^{p_\zeta} \cup d^{p_\xi} \cup d^*$ are rational.

Let us verify that d^t extends $d^{p_\zeta} \cup d^{p_\xi} \cup d^*$. Suppose $x \in u_\zeta, y \in u_\xi$ and $d^*(x, y)$ is defined. Any path from x to y other than (x, y) must contain some edge with a distance in $(u^{p_\zeta}, d^{p_\zeta})$, and all those distances are much larger than $d^*(x, y)$, so (x, y) is the shortest path from x to u. Suppose now that $x, y \in u^{p_\zeta}$. So (x, y) is the shortest path among all paths that lie in u^{p_ζ} and the path of minimal length from x to y among all paths that contain at least one d^* edge is necessarily the path (x, x', y', y) where $x' = g_{\zeta,\xi}(x)$ and $y' = g_{\zeta,\xi}(y)$ (since (u^{p_ξ}, d^{p_ξ}) is a metric space). Since $d^{p_\zeta}(x, y) = d^{p_\xi}(x', y')$ by condition (3), the length of this path is larger than $d^{p_\zeta}(x, y)$.

Let $f_\alpha^t = f_\zeta^{p_\zeta} \cup f_\zeta^{p_\xi}$, where we formally take f_α^p to be the empty function if $\alpha \notin w^p$.

Let $w^t = w^{p_\zeta} \cup w^{p_\xi}$ and let $\varepsilon_\alpha^t = \max\{\varepsilon_\alpha^{p_\zeta}, \varepsilon_\alpha^{p_\xi}\}$ where ε_α^p is taken as 0 if $\alpha \notin w^p$ (recall that $\varepsilon_\alpha^{p_\zeta} = \varepsilon_\alpha^{p_\xi}$ for $\alpha \in w$).

Now t is defined, and extends both p_ζ and p_ξ; one only needs to verify that $t \in Q$. For that, we need to verify that each f_α^t is $(1 + \varepsilon_\alpha^t)$-bi-Lipschitz. For $\alpha \notin w$ this is trivial. Suppose $\alpha \in w$ and $i, h \in \mathrm{dom} f_\alpha^t = r_\alpha \cup s_\zeta \cup s_\xi$. The only case to check is when $i \in s_\alpha^\zeta$ and $j \in s_\alpha^\xi$. If $h_\alpha^{\zeta,\xi}(i) = j$ then this is taken care of by the choice of $d^t(f_\alpha^{p_\zeta}(i), f_\alpha^{p_\xi}(j)) = d^*(f_\alpha^{p_\zeta}(i), f_\alpha^{p_\xi}(j))$.

We are left with the main case: $j \neq h_\alpha^{\zeta,\xi}(i)$. Let $x = f_\alpha^{p_\zeta}(i)$, $y = f_\alpha^{p_\xi}(j)$ and denote $a' = d_\alpha(i,j)$, $b' = d^t(x,y)$. Let $j' \in x^\zeta_\alpha$ be such that $h^{\zeta,\xi}_\alpha(j') = j$ and let $y' = g^{-1}_{\zeta,\xi}(y)$ (so $f^{p_\zeta}_\zeta(j') = y'$).

Denote $b = d^t(x,y')$, $a = d_\alpha(i,j')$. We have that $a/\lambda < b < \lambda a$, and need to prove $a'/\lambda < b' < \lambda b'$.

We have $d_\alpha(j,j') < \delta \le \delta_1/2$, hence $d^t(y,y') < \delta_1$. By the triangle inequality in u^t, we have $|b - b'| < \delta_1$. On the other hand, by the triangle inequality in (ω_1, d_α), we have $|a' - a| < \delta < \delta_1$. Thus, by the choice of δ_1 so that (***iii***) holds if (***ii***) holds and $|a - a'| < \delta_1$, $|b - b'| < \delta_1$, we have $a'/\lambda < b' < \lambda a'$, as required. ∎

Let $P = \langle P_\beta, Q_\beta : \beta < \omega_2\rangle$ be a finite support iteration of length ω_2 in which each factor Q_β is the forcing notion we defined above, in V^{P_β}. Since each Q_β satisfies the ccc, the whole iteration satisfies the ccc and no cardinals or cofinalities are collapsed on the way — in particular, the collection $\{A_\alpha : \alpha < \omega_3\}$ required for the definition of Q is preserved.

Since every metric d on ω_1 appears in some intermediate stage, the universe V^P satisfies that there is a collection of ω_2 separable metrics on ω_1 so that every separable metric on ω_1 is almost isometrically embedded into one of them.

There is no particular property of ω_1 that was required in the proof. Also, Baumgartner's result holds for other cardinals. We have proved then:

THEOREM 34: *Let $\lambda > \aleph_0$ be a regular cardinal. It is consistent that $2^{\aleph_0} > \lambda^+$ and that there are λ^+ separable metrics on λ such that every separable metric on λ is almost-isometrically embedded into one of them.*

In the next section we shall see that separability is essential for this consistency result for all regular $\lambda > \aleph_1$.

5.2. NEGATIVE UNIVERSALITY RESULTS FOR GENERAL METRIC SPACES BELOW THE CONTINUUM. Now we show that the consistency proved in the previous section for separable metric spaces of regular cardinality $\lambda < 2^{\aleph_0}$ is not possible for metric spaces in general if $\lambda > \aleph_1$. Even a weaker fact fails: there cannot be fewer than continuum metric spaces on λ so that every metric space is bi-Lipschitz embeddable into one of them if $\aleph_1 < \lambda < 2^{\aleph_0}$ and λ is regular.

The technique we use here is associating invariants to a structure modulo a club-guessing ideal. This technique was introduced in [10] and used there in the context of linear orders. The same technique has been used since for other classes of structure as well — models of stable theories, Abelian groups, infinite graphs and Eberlein compacta [10, 11, 12, 4, 9, 5].

We shall prove:

THEOREM 35: *If $\aleph_1 < \lambda < 2^{\aleph_0}$ and λ is a regular cardinal, then for every $\kappa < 2^{\aleph_0}$ and metric spaces $\{(\lambda, d_i) : i < \kappa\}$ there exists an ultra-metric space (λ, d) that is not bi-Lipschitz embeddable into (λ, d_i) for all $i < \kappa$. In particular, there is no single metric space (λ_2, d) into which every ultra-metric space of cardinality λ is bi-Lipschitz embedded.*

The proof of Theorem 35 uses the preservation and construction of invariants. We start with preservation.

Let $\lambda > \aleph_1$ be a regular cardinal. Let $S_0^\lambda = \{\delta < \lambda : \mathrm{cf}\delta = \omega\}$, the stationary subset of λ of countably cofinal elements. For a regular $\lambda > \aleph_1$ we may fix, for the rest of the section, a club guessing sequence $\overline{C} = \langle c_\alpha : \delta \in S_0^\lambda \rangle$ [17, 10]:

(1) $c_\delta \subseteq \delta = \sup c_\delta$ and $\mathrm{otp} c_\delta = \omega$ for all $\delta \in S_0^\lambda$;

(2) for every club $E \subseteq \lambda$ the set $S(E) := \{\delta \in S_0^\lambda : c_\delta \subseteq E\}$ is stationary.

For each $\delta \in S_0^\lambda$ let $\langle \alpha_n^\delta : n < \omega \rangle$ be the increasing enumeration of c_δ.

Let (λ, d) be a given metric space. Let X_α denote the subspace $\{\gamma : \gamma < \alpha\}$. Let $d(\beta, X_\alpha)$ denote the distance of β from X_α, that is, the infimum over all $d(\beta, \gamma)$ for $\gamma < \alpha$. Thus, if $\langle \alpha_n : n \in \mathbb{N} \rangle$ is increasing and $\alpha_n < \beta$ for all n, then $\langle d(\beta, X_{\alpha_n}) : n < \omega \rangle$ is a weakly decreasing sequence of nonnegative real numbers.

LEMMA 36: *Suppose (λ, d_i) are metric spaces for $i = 1, 2$ and $f \colon \lambda \to \lambda$ is a K-bi-Lipschitz embedding of (λ, d_1) in (λ, d_2) with constant $K \geq 1$. Then there is a club $E \subseteq \lambda$ such that: for all $\alpha \in E$ and $\beta > \alpha$ we have $f(\beta) > \alpha$ and*

$$(1) \qquad (2K)^{-1} d_1(\beta, X_\alpha) \leq d_2(f(\beta), X_\alpha) \leq K d_1(\beta, X_\alpha).$$

Proof: Consider the structure $\mathcal{M} = (\lambda; d_1, d_2, f, \langle P_q : q \in \mathbb{Q}^+ \rangle)$ where P_q is a binary predicate so that $\mathcal{M} \models P_q(\beta_1, \beta_2)$ iff $d_2(\beta_1, \beta_2) < q$.

Let $E = \{\alpha < \lambda : M{\restriction}X_\alpha \prec M\}$. Then $E \subseteq \lambda$ is a club. Suppose that $\alpha \in E$. Then X_α is closed under f and f^{-1} and therefore $f(\beta) > \alpha$. The inequality $d_2(f(\beta), X_\alpha) \leq K d_1(\beta, X_\alpha)$ is clear because X_α is preserved under f. For the other inequality suppose that $\gamma \in X_\alpha$ is arbitrary and let $\varepsilon := d_2(f(\beta), \gamma)$. Let $q > \varepsilon$ be an arbitrary rational number. Now $\mathcal{M} \models P_q(f(\beta), \gamma)$ and therefore, by $\mathcal{M}{\restriction}X_\alpha \prec \mathcal{M}$, there exists some $\beta' \in X_\alpha$ so that $\mathcal{M} \models P_q((f(\beta'), \gamma)$. Thus

$$d_2(f(\beta), f(\beta')) \leq d_2(f(\beta), \gamma) + d_2(\gamma, f(\beta')) < 2q.$$

Since $K^{-1} d_1(\beta, \beta') \leq d_2(f(\beta), f(\beta'))$, it follows that $K^{-1} d_1(\beta, X_\alpha) < 2q$, and since $q > \varepsilon$ is an arbitrary rational, it follows that $(2K)^{-1} d_1(\beta, X_\alpha) \leq d_2(f(\beta), X_\alpha)$. ∎

LEMMA 37: *Suppose (λ, d_i) are metric spaces for $i = 1, 2$, that $K \geq 1$ and that $f\colon \lambda \to \lambda$ is a K-bi-Lipschitz embedding of (λ, d_1) into (λ, d_2). Let $E \subseteq \lambda$ be a club as guaranteed by the previous lemma. Suppose $\delta \in S(E)$, $\beta > \delta$ and $\langle \alpha_n^\delta : n < \omega\rangle$ is the increasing enumeration of c_δ. Let $\varepsilon_n = d_1(\beta, X_{\alpha_n^\delta})$ and let $\varepsilon'_n = d_2(f(\beta), X_{\alpha_n^\delta})$. Then for each n,*

$$(2) \qquad (2K^2)^{-1}\varepsilon_n/\varepsilon_{n+1} \leq \varepsilon'_n/\varepsilon'_{n+1} \leq 2K^2\varepsilon_n/\varepsilon_{n+1}.$$

Proof: The proof follows immediately from (1). ∎

We show now how to code a real (a subset of ω) in a metric space and retrieve it from a larger metric space in which the former space is bi-Lipschitz embedded. The idea is quite simple: the real is coded as the set of places α_n^δ for which the distance $d(\beta, X_{\alpha_{n+1}^\delta})$ decreases dramatically compared to $d(\beta, X_{\alpha_n^\delta})$, for some element β in the space. If the decrease in the distance is sufficiently large compared to the Lipschitz constant, then it is still possible to recognize the set of n-s at which the distance decreases also in the larger space, in spite of the distortion caused by the embedding.

Definition 38: Suppose $\lambda > \aleph_1$ is regular, $\bar{c} = \langle c_\delta : \delta \in S_0^\lambda\rangle$ is a club-guessing sequence and (λ, d) is metric space. For $\delta \in S_0^\lambda$ let $\langle \alpha_n : n \in \omega\rangle$ be the increasing enumeration of c_δ. Let $\beta \geq \lambda$ be some ordinal in λ and denote $\varepsilon_n := d(\beta, X_{\alpha_n})$.

For a subset $A \subseteq \omega$ we write:

(1) $\Phi_d(\beta, \delta, A, K)$ if, for every $n \in \omega$, either $\varepsilon_n/\varepsilon_{n+1} = 1$ or $\varepsilon_n/\varepsilon_{n+1} > 4K^4$ and $A = \{n : \varepsilon_n/\varepsilon_{n+1} > 4K^4\}$.

(2) $\Theta_d(\beta, \delta, A, K)$ if $A = \{n : \varepsilon_n/\varepsilon_{n+1} > 2K^2\}$.

From (2) there follows:

LEMMA 39 (Preservation Lemma): *Suppose λ is as in the previous definition, d_1 and d_2 are metrics on λ, $K \geq 1$ and $f\colon \lambda \to \lambda$ is a K-bi-Lipschitz embedding of (λ, d_1) in (λ, d_2). Then there is a club $E \subseteq \lambda$ such that for all $\delta \in S(E)$, $A \subseteq \omega$ and $\beta > \delta$, we have $f(\beta) > \delta$ and $\Phi_{d_1}(\beta, \delta, A, K) \Rightarrow \Theta_{d_2}(f(\beta), \delta, A, K)$.*

LEMMA 40 (Construction Lemma): *For every regular cardinal $\lambda > \aleph_1$ and infinite $A \subseteq \omega$ there is an ultra-metric space (λ, d) and a club $E \subseteq \lambda$ so that, for all $\delta \in S(E)$ and integer $K \geq 1$, there exists $\beta > \delta$ with $\Phi_d(\beta, \delta, A, K)$.*

Proof: Suppose $A \subseteq \omega$ is infinite and let $\langle a_n : n < \omega\rangle$ be the increasing enumeration of A. Let λ^ω be the tree of all ω-sequences over λ and let d be the metric so that for distinct $\eta_1, \eta_2 \in (\omega_2)^\omega$, $d(\eta_1, \eta_2) = 1/(n+1)$, where n

is the length of the largest common initial segment of η_1, η_2 (or, equivalently, $d(\eta_1, \eta_2) = 1/(|\eta_1 \cap \eta_2| + 1)$). Every *finite* sequence $t: n \to \lambda$ determines a basic clopen ball of radius $1/(n+1)$ in (λ^ω, d), which is $B_t = \{\eta \in \lambda^\omega : t \subseteq \eta\}$.

By induction on $\alpha < \lambda$ define an increasing and continuous chain of subsets $X_\alpha \subseteq \lambda^\omega$ so that:

(1) $|X_\alpha| < \lambda$ for all $\alpha < \lambda$.

(2) $X_\alpha \subseteq X_{\alpha+1}$ and $X_\alpha = \bigcup_{\beta<\alpha} X_\beta$ if $\alpha < \lambda$ is limit.

(3) For every $\nu \in X_\alpha$ and $k < \omega$ there exists $\eta \in X_{\alpha+1} \setminus X_\alpha$ so that $\nu{\restriction}k \subseteq \eta$ but $B_{\nu\restriction(k+1)} \cap X_\alpha = \emptyset$.

(4) If $\alpha = \delta \in S^\lambda_0$, $\langle \alpha_n : n < \omega\rangle$ is the increasing enumeration of c_δ, then for every integer $K \geq 1$ a sequence ν_K is defined as follows. Let $b_n = (4K^2)^{n+1}$. Define an increasing sequence of finite sequences t_n with $|t_n| = b_n$ by induction on n as follows: $t_0 = \langle\rangle$. Suppose that t_n is defined and $B_{t_n} \cap X_{\alpha_{a_n}} \neq \emptyset$. Choose $\eta \in X_{\alpha_{a_n}+1} \cap B_{t_n}$ so that $B_{\eta\restriction(b_{n+1})} \cap X_{\alpha_{a_n}} = \emptyset$. Now let $t_{n+1} = \eta{\restriction}(b_{n+1})$.

Finally, let $\nu_K = \bigcup_n t_n$. Put ν_k in $X_{\delta+1}$ for each integer $K \geq 1$.

There is no problem to define X_α for all $\alpha < \lambda$. Let $X = \bigcup_{\alpha<\lambda} X_\alpha$. Fix a 1-1 onto function $F: X \to \lambda$ and let d be the metric on ω_2 which makes F an isometry. Observe that for some club $E \subseteq \lambda$, $F[X_\alpha] = \alpha$ for all $\alpha \in E$. If $c_\delta \subseteq E$, let ν_K be the sequence we put in $X_{\alpha+1}$ in clause (4) of the inductive definition and let $\beta_K = F(\nu_K)$. We leave it to the reader to verify that $\Phi_d(\beta_K, \delta, A, K)$. ∎

Now we arrive at:

Proof of Theorem 35: The proof of the theorem follows from both lemmas. Suppose $\kappa < 2^{\aleph_0}$ and d_i is a metric on λ for each $i < \kappa$. Consider the set

$$\mathcal{A} := \{A \subseteq \omega : (\exists i < \kappa)(\exists \delta \in S^\lambda_0)(\exists K \in \omega \setminus \{0\})(\exists \beta > \delta)[\Theta_{d_i}(\beta, \delta, A, K)]\}.$$

This set has cardinality $\leq \lambda \cdot \kappa < 2^{\aleph_0}$. Therefore, there is some infinite $A \in \mathcal{P}(\omega) \setminus \mathcal{A}$. By Lemma 40 there is some ultra-metric d on λ and a club $E \subseteq \lambda$ so that for all $\delta \in S^\lambda_0$ and integer $K \geq 1$ there is $\beta > \delta$ with $\Phi_d(\beta, \delta, A, K)$. If, for some $i < \kappa$ and integer $K \geq 1$, there is a K-bi-Lipschitz embedding φ of (λ, d) into (λ, d_i), then from Lemma 39 there is some club $E' \subseteq \lambda$ so that for all $\delta \in S(E)$, $\beta > \delta$ and infinite $A \subseteq \omega$, $\varphi(\beta) > \delta$ and $\Phi_d(\beta, \delta, A, K) \Rightarrow \Theta_{d_i}(\beta, \delta, A, K)$. Let $E'' = E \cap E'$. Choose $\delta \in S(E'')$. Now for some $\beta > \delta$ we have $\Phi_d(\beta, \delta, A)$, hence $\Theta_{d_i}(\beta, \delta, A, K)$ — contrary to the choice of $A \notin \mathcal{A}$. ∎

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